Extended E_8 invariance of 11-dimensional supergravity

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ABSTRACT: The hyperbolic Kac-Moody algebra E_{10} has repeatedly been suggested to play a crucial role in the symmetry structure of M-theory. Recently, following the analysis of the asymptotic behaviour of the supergravity fields near a cosmological singularity, this question has received a new impulse. It has been argued that one way to exhibit the symmetry was to rewrite the supergravity equations as the equations of motion of the non-linear sigma model $E_{10}/K(E_{10})$. This attempt, in line with the established result that the scalar fields which appear in the toroidal compactification down to three spacetime dimensions form the coset $E_8/SO(16)$, was verified for the first bosonic levels in a level expansion of the theory. We show that the same features remain valid when one includes the gravitino field.

KEYWORDS: Supergravity Models, M-Theory.

Contents

1. Introduction

The hyperbolic Kac-Moody algebra E_{10}, whose Dynkin diagram is given in Fig.1, has repeatedly been argued to play a crucial role in the symmetry structure of M-theory [1–3].

This infinite-dimensional algebra has a complicated structure that has not been deciphered yet. In order to analyse further its root pattern, it was found convenient in [4] to introduce a "level" for any root α, defined as the number of times the simple root α_0 occurs in the decomposition of α.

The roots α_1 through α_9 define a subalgebra $sl(10)$. Reflections in these roots define the finite Weyl group W_{A_9} ($\simeq S_{10}$) of A_9, which acts naturally on the roots of E_{10}. If we express the roots of E_{10} in terms of the spatial scale factors β^i appearing naturally in cosmology [5], the action of W_{A_9} is simply to permute the β's. The level is invariant under W_{A_9}. Consider the set R_{E_8} of roots of the E_8 subalgebra associated with the simple roots α_0 through α_7. By acting with W_{A_9} on R_{E_8}, one generates a larger set $\tilde{R}_{E_8}$ of roots. This set will be called the extended set of roots of E_8. By construction, the roots in $\tilde{R}_{E_8}$ are all
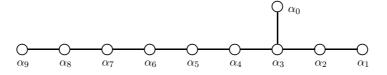

Figure 1: The Dynkin diagram of E_{10}.

real and have length squared equal to 2. There is an interesting description of the roots in $\tilde{R}_{E_8}$ in terms of the level. One can easily verify that all the roots at level 0, ± 1 and ± 2, as well as all the *real* roots at level ± 3 exhaust $\tilde{R}_{E_8}$. This includes, in particular, all the roots with $|\text{height}| < 30$.

Recently, following the analysis *à la* BKL [6, 7] of the asymptotic behaviour of the supergravity fields near a cosmological singularity, the question of the hidden symmetries of eleven-dimensional supergravity has received a new impulse [5]. It has been argued that one way to exhibit the symmetry was to rewrite the supergravity equations as the equations of motion of the non-linear sigma model $E_{10}/K(E_{10})$ [4].

The first attempt for rewriting the equations of motion of eleven-dimensional supergravity as non-linear sigma model equations of motion – in line with the established result that the scalar fields which appear in the toroidal compactification down to three spacetime dimensions form the coset $E_8/SO(16)$ [8] — is due to [9]. In that approach, it is the larger infinite-dimensional algebra E_{11} which is priviledged. Various evidence supporting E_{11} was provided in [9, 10]. Here, we shall stick to (the subalgebra) E_{10}, for which the dynamical formulation is clearer.

The idea of rewriting the equations of motion of eleven-dimensional supergravity as equations of motion of $E_{10}/K(E_{10})$ was verified in [4] for the first bosonic levels in a level expansion of the theory. More precisely, it was verified that in the coset model $E_{10}/K(E_{10})$, the fields corresponding to the Cartan subalgebra and to the positive roots $\in \tilde{R}_{E_8}$ have an interpretation in terms of the (bosonic) supergravity fields ("dictionary" of [4]). Furthermore, there is a perfect match of the supergravity equations of motion and the coset model equations of motion for the fields corresponding to these real roots. This extended E_8-invariance, which combines the known E_8-invariance and the manifest sl_{10}-invariance, is a first necessary step in exhibiting the full E_{10} symmetry. Further indication on the meaning of the fields associated with the higher roots in terms of gradient expansions, using partly information from E_9, was also given in [4].

The purpose of this paper is to explicitly verify the extended E_8-invariance of the fermionic sector of 11-dimensional supergravity. This amounts to showing that up to the requested level, the fermionic part of the supergravity Lagrangian, which is first order in the derivatives, can be written as

$$i\Psi^T M D_t \Psi \tag{1.1}$$

where (i) Ψ is an infinite object that combines the spatial components of the gravitino field ψ_a and its successive gradients

$$\Psi = (\psi_a, \cdots) \tag{1.2}$$

in such a way that Ψ transforms in the representation of $K(E_{10})$ that reduces to the spin $3/2$ representation of $SO(10)$; (ii) M is a $K(E_{10})$-invariant (infinite) matrix; and (iii) D_t is the $K(E_{10})$ covariant derivative. (We work in the gauge $\psi'_0 = 0$, where ψ'_0 is the redefined temporal component of the gravitino field familiar from dimensional reduction [8],

$$\psi'_0 = \psi_0 - \gamma_0 \gamma^a \psi_a, \tag{1.3}$$

so that the temporal component of ψ_μ no longer appears.)

In fact, for the roots considered here, one can truncate Ψ to the undifferentiated components ψ_a. The next components – and the precise dictionary yielding their relationship with the gravitino field gradients – are not needed. In view of the fact that the undifferentiated components of the gravitino field form a representation of the maximal compact subgroup $SO(16)$ of E_8 in the reduction to three dimensions, without the need to introduce gradients or duals, this result is not unexpected.

The crux of the computation consists in constructing the representation of $K(E_{10})$ up to the required level. This is done in the next section, where we compare and contrast the spin 1/2 and spin 3/2 representations of $K(E_{10})$. The technically simpler case of the spin 1/2 representation was investigated in [11], where it was shown that the Dirac Lagrangian was compatible with extended E_8 invariance provided one introduces an appropriate Pauli coupling with the 3-form. Our work overlaps the work [12] on the fermionic representations of $K(E_{11})$ as well as the analyses of [13] on the maximal compact subgroups of $E_{n,n}$ and of [14] on $K(E_9)$.

We then investigate the conjectured infinite-dimensional symmetry E_{10} of the Lagrangian of [15]. We find that the fermionic part also takes the form dictated by extended E_8-invariance, with the correct covariant derivatives appearing up to the appropriate level. As observed by previous authors and in particular in [11], there is an interesting interplay between supersymmetry and the hidden symmetries.

2. 'Spin 3/2' Representation of $K(E_{10})$

2.1 Level 0

To construct the 'spin 3/2' representation of $K(E_{10})$, we have to extend the level 0 part which is the usual $SO(10)$ 'spin 3/2' parametrized by a set of 10 spinors χ_m, where $m = 1 \ldots 10$ is a space index. (The level is not a grading for KE_{10} but a filtration, defined modulo lower order terms.) The $so(10)$ generators k^{ij} act on χ_m as

$$k^{ij} . \chi_m = \frac{1}{2} \gamma^{ij} \chi_m + \delta_m^{\ i} \chi^j - \delta_m^{\ j} \chi^i . \tag{2.1}$$

The aim is to rewrite the Rarita-Schwinger term with all couplings of the fermionic field, up to higher order fermionic terms, into the form (1.1).

2.2 Level 1

Beyond $SO(10)$, the first level couples to F_{0abc}. To reproduce the supergravity Lagrangian, the level 1 generators must contain products of γ matrices where the number of matrices is odd and at most five. Indeed, the matrix M in (1.1) is proportional to the antisymmetric product γ^{ab} (as one sees by expanding the supergravity Lagrangian $\mathcal{L} \sim i\psi_a^T \gamma^{ab} \dot{\psi}_b + \cdots$), while F_{0abc} is coupled to fermions, in the supergravity Lagrangian, through terms $\psi_m^T \gamma^{mabcn} \psi_n$, $\psi_a^T \gamma^{bc} \gamma^n \psi_n$ and $\psi_a^T \gamma^b \psi_c$. In addition, the generators must be covariant with respect to $SO(10)$. This gives the general form

$$k^{abc} . \chi_m = A \gamma_m^{\ nabc} \chi_n + 3B \delta_m^{[a} \gamma^{bc]n} \chi_n + 3C \gamma_m^{[ab} \chi^{c]} + 6D \delta_m^{[a} \gamma^b \chi^{c]} + E \gamma^{abc} \chi_m \tag{2.2}$$

where A, B, C, D, E are constants to be fixed. In fact, it is well known that such generators do appear in the dimensional reduction of supergravity. If d dimensions are reduced, the generators mix only χ_m with $1 \leq m \leq d$. Therefore we set the terms involving summation on n on the right hand side of (2.2) equal to zero: $A = B = 0$.

To fix the coefficients C, D and E, we must check the commutations relations. Commutation with level 0 generators is automatic, as (2.2) is covariant with respect to $SO(10)$. What is non-trivial is commutation of the generators at level 1 with themselves. The generators K^{abc} of $K(E_{10})$ at level 1 fulfill

$$[K^{abc}, K^{def}] = K^{abcdef} - 18\delta^{ad}\delta^{be}K^{cf} \tag{2.3}$$

(with antisymmetrization in (a, b, c) and (b, c, d) in $\delta_{ad}\delta_{be}K_{cf}$) as it follows from the E_{10} commutation relations. In order to have a representation of $K(E_{10})$, the k^{abc} must obey the same algebra,

$$[k^{abc}, k^{def}] = k^{abcdef} - \delta^{ad}\delta^{be}k^{cf}. \tag{2.4}$$

When all the indices are distinct, (2.4) defines the generators at level 2. One gets non trivial constraints when two or more indices are equal. Namely, there are two relations which must be imposed:

$$\left[k^{abc}, k^{abd}\right] = -k^{cd} \tag{2.5}$$

$$\left[k^{abc}, k^{ade}\right] = 0 \tag{2.6}$$

where different indices are supposed to be distinct. In fact, as we shall discuss in the sequel, all other commutation relations which have to be checked for higher levels can be derived from this ones using the Jacobi identity. One can verify (2.5) and (2.6) directly or using *FORM* [16]. One finds that these two relations are satisfied if and only if

$$C = -\frac{1}{3}\epsilon, \quad D = \frac{2}{3}\epsilon, \quad E = \frac{1}{2}\epsilon \tag{2.7}$$

with $\epsilon = \pm 1$. In fact one can change the sign of ϵ by reversing the signs of all the generators at the odd levels. This does not change the algebra. We shall use this freedom to set $\epsilon = 1$ in order to match the conventions for the supergravity Lagrangian. Putting everything together, the level 1 generator is

$$k^{abc}.\chi_m = \frac{1}{2}\gamma^{abc}\chi_m - \gamma^{m[ab}\chi^{c]} + 4\delta_m^{[a}\gamma^b\chi^{c]} . \tag{2.8}$$

2.3 Level 2

The expression just obtained for the level 1 generators can be used to compute the level 2 generator

$$k^{abcdef} = \left[k^{abc}, k^{def}\right] \tag{2.9}$$

which is totally antisymmetric in its indices, as it can be shown using the Jacobi identity. Explicitly, Eq.(2.8) gives

$$k^{abcdef}.\chi_m = \frac{1}{2}\gamma^{abcdef}\chi_m + 4\gamma_m^{[abcde}\chi^{f]} - 10\delta_m^{[a}\gamma^{bcde}\chi^{f]} . \tag{2.10}$$

2.4 Level 3

We now turn to level 3. There are two types of roots. Real roots have generators

$$k^{a;abcdefgh} = \left[k^{abc}, k^{adefgh} \right] \tag{2.11}$$

(without summation on a and all other indices distinct). They are easily computed to act as

$$k^{a;abcdefgh} \cdot \chi_m = \frac{1}{2} \gamma^{bcdefgh} \chi_m + 2\gamma_m{}^{abcdefgh} \chi^a + 16\delta_m^a \gamma^{[abcdefg} \chi^{h]} - 7\gamma_m{}^{[bcdefg} \chi^{h]} \, . \tag{2.12}$$

In addition, there are generators $k^{a;bcdefghi}$ with all indices distinct, corresponding to null roots. From

$$\left[k^{abc}, k^{defghi} \right] = 3k^{[a;bc]defghi} \tag{2.13}$$

(with all indices distinct) one finds

$$k^{a;bcdefghi} \cdot \chi_m = -2 \left(\gamma_m{}^{[abcdefgh} \chi^{i]} - \gamma_m{}^{bcdefghi} \chi^a \right) - 16 \left(\delta_m^{[a} \gamma^{bcdefgh} \chi^{i]} - \delta_m^a \gamma^{[bcdefgh} \chi^{i]} \right) \, . \tag{2.14}$$

Combining these results, one finds that the level 3 generators can be written as

$$\begin{aligned} k^{a;bcdefghi} \cdot \chi_m = &-2 \left(\gamma_m{}^{[abcdefgh} \chi^{i]} - \gamma_m{}^{bcdefghi} \chi^a \right) - 16 \left(\delta_m^{[a} \gamma^{bcdefgh} \chi^{i]} - \delta_m^a \gamma^{[bcdefgh} \chi^{i]} \right) \\ &+ 4\delta^{a[b} \gamma^{cdefghi]} \chi_m - 56\gamma^{m[bcdefg} \delta^{\hat{a}h} \chi^{i]} \, . \end{aligned} \tag{2.15}$$

(where the hat over a means that it is not involved in the antisymmetrization). Note that if one multiplies the generator (2.15) by a parameter $\mu_{a;bcdefghi}$ with the symmetries of the level 3 Young tableau (in particular, $\mu_{[a;bcdefghi]} = 0$), the first terms in the two parentheses disappear. Furthermore, the totally antisymmetric part of the full level 3 generator vanishes. The condition $\mu_{[a;bcdefghi]} = 0$ on $\mu_{a;bcdefghi}$ is equivalent to the tracelessness of its dual.

2.5 Compatibility checks

Having defined the generators of the 'spin 3/2' representation up to level 3, we must now check that they fulfill all the necessary compatibility conditions expressing that they represent the $K(E_{10})$ algebra up to that level (encompassing the compatibility conditions (2.5) and (2.6) found above). This is actually a consequence of the Jacobi identity and of the known $SO(16)$ invariance in 3 dimensions, as well as of the manifest spatial $SL(10)$ covariance that makes all spatial directions equivalent.

Consider for instance the commutators of level 1 generators with level 2 generators. The $K(E_{10})$ algebra is

$$[K^{abc}, K^{defghi}] = 3K^{[a;bc]defghi} - 5!\delta^{ad}\delta^{be}\delta^{cf} K^{ghi} \tag{2.16}$$

Thus, one must have

$$[k^{abc}, k^{defghi}] = 3k^{[a;bc]defghi} - 5!\delta^{ad}\delta^{be}\delta^{cf} k^{ghi} \tag{2.17}$$

These relations are constraints on k^{abc} and k^{defghi} when the level 3 generators are absent, which occurs when (at least) two pairs of indices are equal. But in that case, there are only (at most) 7 distinct values taken by the indices and the relations are then part of the known $SO(16)$ invariance emerging in 3 dimensions. In fact, the relations (2.17) are known to hold when the indices take at most 8 distinct values, which allows $k^{a;acdefghi}$ with a pair of repeated indices. These 8 values can be thought of as parameterizing the 8 transverse dimensions of the dimensional reduction. Note that since the index m in (2.8) can be distinct from the 8 "transverse" indices, we have both the 'spin 1/2' and the 'spin 3/2' (i.e., the vector and the spinor) representations of $SO(16)$, showing the relevance of the analysis of [11] in the present context.

Similarly, the commutation of two level 2 generators read

$$[K^{abcdef}, K^{ghijkl}] = -6 \cdot 6! \delta^{ag} \delta^{bh} \delta^{ci} \delta^{dj} \delta^{ek} K^{fl} + \text{ "more"} \tag{2.18}$$

where "more" denotes level 4 generators. Thus, one must have

$$[k^{abcdef}, k^{ghijkl}] = -6 \cdot 6! \delta^{ag} \delta^{bh} \delta^{ci} \delta^{dj} \delta^{ek} k^{fl} + \text{ "more"} \tag{2.19}$$

These relations are constraints when the level 4 generators are absent[1]. Now, the level 4 generators are in the representation (001000001) characterized by a Young tableau with one 9-box column and one 3-box column, and in the representation (200000000) characterized by a Young tableau with one 10-box column and two 1-box columns [17, 18]. To get rid of these level 4 representations, one must again assume that the indices take at most 8 distinct values to have sufficiently many repetitions. [If one allows 9 distinct values, one can fill the tableau (001000001) non trivially.] But then, $SO(16)$ "takes over" and guarantees that the constraints are fulfilled. The same is true for the commutation relations of the level 1 generators with the level 3 generators, which also involve generically the level 4 generators unless the indices take only at most 8 distinct values (which forces in particular the level-3 generators to have one repetition, i.e., to correspond to real roots).

Finally, the level 5 generators and the level-6 generators, which occur in the commutation relations of level 2 with level 3, and level 3 with itself, involve also representations associated with Young tableaux having a column with 9 or 10 boxes [17, 18]. For these to be absent, the indices must again take on at most 8 distinct values. The commutation relations reduce then to those of $SO(16)$, known to be valid.

2.6 'Spin 1/2' representation

We note that if one keeps in the above generators (2.1), (2.8), (2.10) and (2.15) only the terms in which the index m does not transform, one gets the 'spin 1/2' representation investigated in [11]. A notable feature of that representation is that it does not see the level-3 generators associated with imaginary roots, as one sees from (2.14).

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[1]When the level 4 generators are present, the relations (2.19) are consequences of the definition of the level 4 generators – usually defined through commutation of level 1 with level 3 –, as a result of the Jacobi identity.

It should be stressed that up to level 3, the commutation relations of the $K(E_n)$ subgroups are all very similar for $n \geq 8$ ([12–14]). A more complete analysis of the 'spin 3/2' and 'spin 1/2' representations of $K(E_9)$ will be given in [19].

3. Extended E_8 Invariance of Supergravity Lagrangian

The fermionic part of 11-dimensional supergravity is

$$e^{(11)}(-\frac{1}{2}\overline{\psi}_\mu\gamma^{\mu\rho\nu}D_\rho\psi_\nu - \frac{1}{96}\overline{\psi}_\mu\gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu F_{\alpha\beta\gamma\delta} - \frac{1}{8}\overline{\psi}^\alpha\gamma^{\gamma\delta}\psi^\beta F_{\alpha\beta\gamma\delta})\,, \qquad (3.1)$$

where $e^{(11)}$ is the determinant of the spacetime vielbein and where we have dropped the terms with four fermions. We want to compare this expression with the Lagrangian (1.1), where the $K(E_{10})$ representation is the spin 3/2 one constructed in the previous section. If we expand the Lagrangian (1.1) keeping only terms up to level 3 and using the dictionary of [4] for the $K(E_{10})$ connection, we get (see Eq. (8.7) of [11])

$$-\frac{i}{2}\psi_m^T\gamma^{mn}(\dot{\psi}_n - \frac{1}{2}\omega_{ab}^R k^{ab}.\psi_n - \frac{1}{3!}F_{0abc}k^{abc}.\psi_n - \frac{e}{4!\,6!}\varepsilon_{abcdp_1p_2\cdots p_6}F^{abcd}k_{p_1p_2\cdots p_6}.\psi_n$$
$$-\frac{e}{2.2!\,8!}C^a{}_{rs}\epsilon^{rsbcdefghi}k_{a;bcdefghi}.\psi_n) \ (3.2)$$

where M at this level is given by γ^{mn} and where $\omega_{ab}^R = -\frac{1}{2}(e_a{}^\mu\dot{e}_{\mu b} - e_b{}^\mu\dot{e}_{\mu a})$. In (3.2), e is the determinant of the spatial vielbein, $e^{(11)} = N\,e$ with N the lapse.

We have explicitly checked the matching between (3.2) and (3.1). In order to make the comparison, we

- take the standard lapse N equal to e;

- split the eleven dimensional supergravity Lagrangian (3.1) into space and time using a zero shift ($N^k = 0$) and taking the so-called time gauge for the vielbeins e_μ^a, namely no mixed space-time component;

- rescale the fermions $\psi_n \to e^{1/2}\psi_n$ as in the spin 1/2 case, so that ψ_n in (3.2) is $e^{1/2}\psi_n$ in (3.1);

- take the gauge choice $\psi_0' = 0$ (1.3);

- take the spatial gradient of the fermionic fields equal to zero (these gradients would appear at higher levels);

- assume that the spatial metric is (at that order) spatially homogeneous (i.e., neglect its spatial gradients in the adapted frames) and that the structure constants $C^a{}_{bc} = -C^a{}_{cb}$ of the homogeneity group are traceless (to match the level 3 representation),

$$C^a{}_{ac} = 0.$$

We have also verified that the matrix M is indeed invariant up to that level.

As for the spin 1/2 case, the matching between (3.2) and (3.1) fully covers level 3 under the above condition of tracelessness of C^a_{bc}, including the imaginary roots. For the spin 1/2 case, this is rather direct since the null root part vanishes, but this part does not vanish for the spin 3/2. However, the dictionary of [4] is reliable only for extended E_8.

Finally, we recall that the covariant derivative of the supersymmetry spin 1/2 parameter is also identical with the $K(E_{10})$ covariant derivative up to level 3 [11], so that the supersymmetry transformations are $K(E_{10})$ covariant. The $K(E_{10})$ covariance of the supersymmetry transformations might prove important for understanding the $K(E_{10})$ covariance of the diffeomorphisms, not addressed previously. Information on the diffeormorphisms would follow from the fact that the graded commutator of supersymmetries yields diffeormorphisms (alternatively, the supersymmetry constraints are the square roots of the diffeomorphisms constraints [20]).

4. Conclusions

In this paper, we have shown that the gravitino field of 11-dimensional supergravity is compatible with the conjectured hidden E_{10} symmetry up to the same level as in the bosonic sector. More precisely, we have shown that the fermionic part of the supergravity Lagrangian take the form (1.1) with the correct $K(E_{10})$ covariant derivatives as long as one considers only the connection terms associated with the roots in extended E_8, for which the dictionary relating the bosonic supergravity variables to the sigma-model variables has been established. The computations are to some extent simpler than for the bosonic sector because they involve no dualization. In sigma-model terms, the supergravity action is given by the (first terms of the) action for a spinning particle on the symmetric space $E_{10}/K(E_{10})$, with the internal degrees of freedom in the 'spin 3/2' representation of $K(E_{10})$ (modulo the 4-fermion terms).

This action takes the same form as the action for a Dirac spinor with the appropriate Pauli couplings that make it $K(E_{10})$ covariant [11], where this time the internal degrees of freedom are in the 'spin 1/2' representation. We can thus analyse its dynamics in terms of the conserved $K(E_{10})$ currents along the same lines as in [11] and conclude that the BKL limit holds.

Although the work in this paper is a necessary first step for checking the conjectured E_{10} symmetry, much work remains to be done to fully achieve this goal. To some extent, the analysis remains a bit frustrating because no really new light is shed on the meaning of the higher levels. Most of the computations are controlled by E_8 and manifest sl_{10} covariance. In particular, the imaginary roots, which go beyond E_8 and height 29, still evade a precise dictionary. The works in [21] and in [22] are to our knowledge the only ones where imaginary roots are discussed and are thus particularly precious and important in this perspective.

We have treated explicitly the case of maximal supergravity in this paper, but a similar analysis applies to the other supergravities, described also by infinite-dimensional Kac-Moody algebras (sometimes in non-split forms [23, 24]).

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